

Re-exam

Thermal Physics

2019-2020

Thursday April 9, 2020

8:30-11:30

Due to intervention of the corona virus administrated as an on-line exam on June 25, 2020

Read these instructions carefully before making the exam!

- Write your name and student number on *every* sheet.
- *Make sure to write readable for other people than yourself. Points will NOT be given for answers in illegible writing.*
- *Language; your answers have to be in English.*
- Use *separate sheets of paper* for your answers to problems 1, 2, 3 and for problems 4, 5, 6 (see figure at the bottom of this page).
- Use of a (graphing) calculator is allowed.
- This exam consists of 6 problems.
- The weight of the problems is: Problem 1 (P1=15 pts); Problem 2 (P2=15 pts); Problem 3 (P3=15 pts); Problem 4 (P4=15 pts) Problem 5 (P5=15 pts); Problem 6 (P6=15 pts). Weights of the various subproblems are indicated at the beginning of each problem.
- The grade of the exam is calculated as $(P1+P2+P3+P4+P5+P6+10)/10$.
- For all problems you have to write down your arguments and the intermediate steps in your calculation, *else the answer will be considered as incomplete and points will be deducted.*

PROBLEM 1,2,3 Name S-number		PROBLEM 4,5,6 Name S-number	
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PROBLEM 1

Score: $a+b+c = 5+5+5=15$

- a) Consider throwing with a number of fair dice.

For a system of 2 dice calculate the entropy of throwing:

- a sum of 12,
- a sum of 7.

For a system of 3 dice calculate the entropy of throwing:

- a sum of 18,
- a sum of 10.

Use your result to explain why the concept of entropy is most relevant for very large number of particles.

- b) Under which general circumstances can we approximate the air around us as an ideal gas? Air consists mostly of N_2 . Under which conditions does N_2 have a molar heat capacity $C_{V,m}$ of $3R/2$ and under which conditions does the heat capacity increase to $5R/2$?
- c) Consider two samples, each consisting of 1 mol N_2 being expanded at 0°C from 4 dm^3 to 13.5 dm^3 . Sample 1 expands very slowly in an isothermal fashion, whereas sample 2 expands directly, against a constant external pressure. Determine the expansion work for both cases. Use your result to explain the concept of reversible thermodynamic processes.

PROBLEM 2

Score: $a+b+c+d = 4+4+4+3=15$

The cycle involved in the operation of an internal combustion engine is called the *Otto cycle*. Air can be considered to be the working substance which behaves as an ideal gas. The cycle consists of the following steps:

- 1) a reversible adiabatic compression from A to B;
- 2) a reversible constant-volume pressure increase from B to C due to the combustion of a small amount of fuel;
- 3) a reversible adiabatic expansion from C to D;
- 4) and a reversible and constant-volume pressure decrease from D to A.

Assume that the working substance is one mole of air. In state A, the volume is $V_A = 4$ litre, the pressure is $p_A = 1$ bar, and the temperature is $T_A = 300$ K. It is also known that $V_A = 10 V_B$, $p_C = 5 p_B$, $C_{p,m} = 7/2 R$ and $C_{p,m} - C_{v,m} = R$.

Hint: For adiabatic changes in an ideal gas, we have:

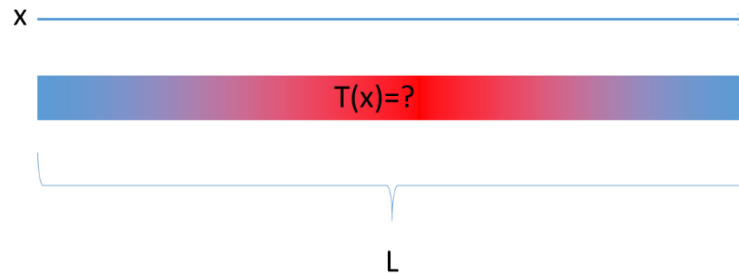
$$\frac{V_i}{V_f} = \left(\frac{T_f}{T_i}\right)^c \text{ with } c = C_{v,m}/R$$

- a) Sketch this thermodynamic cycle in a $p - V$ diagram. Indicate in which steps heat flows and in which direction (into the system and out of the system).
- b) Show that $T_B = 754$ K, $T_C = 3768$ K and $T_D = 1500$ K.
- c) Determine the work done by the system in every step.
- d) The efficiency of this idealized process is $\eta \sim 0.6$. How could you increase the efficiency? Why is the efficiency lower in reality?

PROBLEM 3

Score: $a+b+c=6+6+3=15$

The thermal diffusion equation for a system that includes a heat source, has the form: $\frac{\partial T}{\partial t} = \frac{\kappa}{C} \nabla^2 T + \frac{H}{C}$. Here, $C = \rho c$ is the heat capacity per unit volume, i.e. the material density ρ times the specific heat capacity c . κ is the thermal conductivity of the material and H is the generated heat per unit volume (measured in W/m^3).



Now consider a steel bar of length $L = 1$ m and diameter $D = 1$ mm ($\kappa_{steel} = 50 \text{ Wm}^{-1}\text{K}^{-1}$). Both ends of the bar are connected to a heat bath that ensures the bar ends are kept at the same temperature $T = T_0$. Heat is generated in the bar, by letting an electric current flow through it. After a short while, a *steady state* is reached, in which the temperature profile along the bar does not change anymore.

- Find a functional form for $T(x)$ that is a solution of the thermal diffusion equation for the steady-state situation
- Use the given boundary conditions to determine the integration constants in $T(x)$.
- Determine the temperature in the middle of the metal bar when the bar is heated electrically with 10 W.

PROBLEM 4

Score: $a+b+c+d = 4+4+4+3=15$

Consider a crystal of N independent identical particles. Each of these particles has three non-degenerate energy levels namely, $-\varepsilon$, 0 and ε . The crystal is in equilibrium with a heat bath with temperature T .

- Calculate the partition function Z_1 for a *single* particle.
- Calculate the probabilities P_{-1} , P_0 and P_1 that the energy levels $-\varepsilon$, 0 and ε of this particle are occupied and give the values of P_{-1} , P_0 and P_1 in the low-temperature limit ($T \rightarrow 0$) and the-high temperature limit ($T \rightarrow \infty$).
- Calculate the internal energy U of the *crystal* of N particles.
- Show that the entropy S of the *crystal* of N particles is given by:

$$S = N \left\{ \frac{\varepsilon}{T} \frac{e^{-\beta\varepsilon} - e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} + k \ln(e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}) \right\}$$

and show that this expression is consistent with the third law of thermodynamics.

PROBLEM 5

Score: $a+b+c+d=4+3+4+4=15$

A gas of photons is confined to a cavity with volume V . The cavity is kept at a temperature T and the gas and the cavity are in thermal equilibrium. The single particle (photon) energy levels are $\varepsilon_i = \hbar\omega_i$, $i = 1, 2, \dots$ and the occupation numbers of these energy levels are n_i , $i = 1, 2, \dots$. The partition function Z_{ph} for this gas can be expressed as:

$$Z_{ph} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots e^{-\beta(n_1\varepsilon_1+n_2\varepsilon_2+\dots)}$$

a) Show that the logarithm of this partition function can be written as:

$$\ln Z_{ph} = - \sum_{i=1}^{\infty} \ln(1 - e^{-\beta\varepsilon_i}) \quad \text{with } \varepsilon_i = \hbar\omega_i \text{ the single photon energy levels.}$$

b) Show that density of states of a photon in the cavity can be written as,

$$g(\omega)d\omega = \frac{V\omega^2 d\omega}{\pi^2 c^3}$$

c) Show that the logarithm in a) can be expressed as,

$$\ln Z_{ph} = \frac{\pi^2 k^3}{45 \hbar^3 c^3} VT^3$$

(HINT: transform the sum in an integral, use the density of states from b) and use the table of integrals).

d) Calculate the Helmholtz free energy F , the entropy S , the internal energy U and the pressure P of the photon gas. Express your answers in terms of V , T and physical constants.

PROBLEM 6

Score: $a+b+c+d = 4+4+4+3=15$

Considered a 2D ideal gas of fermions with spin $\frac{1}{2}$ enclosed in a square with area L^2 . The gas is in equilibrium with a heat bath with temperature T and a particle reservoir with chemical potential μ .

- a) Show that the density of states of this 2D ideal gas of electrons is independent of energy and can be written as:

$$g(E)dE = \frac{m_e L^2}{\pi \hbar^2} dE$$

with m_e the electron mass.

We now cool the 2D ideal gas of fermions to temperature $T = 0$.

- b) Give the definition of the Fermi energy E_F .
c) Calculate the Fermi energy for the 2D ideal gas of electrons.
d) Show that at $T = 0$ the internal energy of this gas is given by $U = \frac{1}{2}NE_F$.

Solutions

PROBLEM 1

a)

The entropy is given by $S = k_B \ln \Omega$, thus

2 dice, sum 12: $\Omega = 1$; $S = k_B \ln 1 = 0$

2 dice, sum 7: we have $\{(1,6); (2,5); (3,4)\}$ and their permutations $\{2; 2; 2\}$ giving $\Omega = 2 + 2 + 2 = 6$; $S = k_B \ln 6$

2 dice: total number of possibilities: 36 (6×6)

3 dice, sum 18: $\Omega = 1$; $S = k_B \ln 1 = 0$

3 dice, sum 10: we have $\{(6,2,2); (6,3,1); (5,4,1); (5,2,3); (4,4,2); (4,3,3)\}$ and their permutations $\{3; 6; 6; 6; 3; 3\}$ giving $\Omega = 3 + 6 + 6 + 6 + 3 + 3 = 27$; $S = k_B \ln 27$

3 dice: total number of possibilities: 216 ($6 \times 6 \times 6$)

When going from 2 to 3 dice, the probability for a macrostate (outcome 12, 18) with only a single microstate decreases from $1/36$ to $1/216$. The probability for the macrostate with the most microstates (7, 10) decreases only from $6/36=0.17$ to $27/216=0.125$. For very many dice (particles), virtually only macrostates with the most microstates are populated – entropy is maximized.

b)

Ideal gas: pointlike particles, only repulsive interactions, hard spheres, Newton's laws apply.

This is a good approximation for low pressure (few collisions) and high temperature (attractive interactions don't play an important role). $C_V = \frac{3}{2}R$ represents particles with three degrees of freedom, i.e. translation of the entire molecule in x, y, z . At sufficiently high T , internal degrees of freedom are unlocked, firstly rotation. Rotation along the molecular axis does not contribute much and thus only $\frac{2}{2}R$ is added. As a result, $C_V = \frac{5}{2}R$.

c)

For a reversible isothermal expansion, we have $\bar{d}W = -pdV$ thus,

$$W = - \int p dV$$

And for an ideal gas,

$$W = - \int \frac{nRT}{V} dV$$

$$W = -nRT \int_{V_1}^{V_2} \frac{1}{V} dV = -nRT \ln \frac{V_2}{V_1} = -1 \times 8.3145 \times 273 \ln \frac{13.5}{4} \text{ J} = -2761 \text{ J}$$

For an expansion against constant (the final) pressure we first determine final pressure using the ideal gas law in the end point (volume is 13.5 dm^3) of the expansion,

$$pV = nRT$$

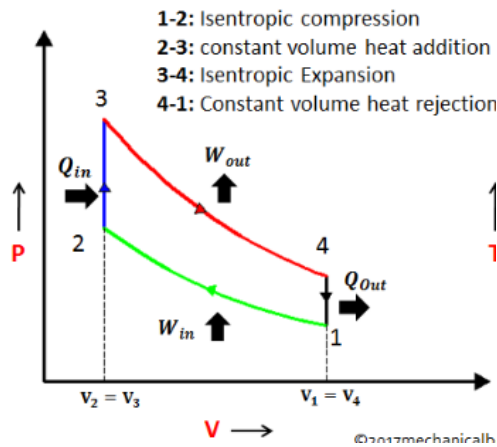
$$p = \frac{nRT}{V} = \frac{1 \times 8.3145 \times 273}{0.0135} \text{ Pa} = 1.68 \times 10^5 \text{ Pa}$$

Then we determine the work,

$$W = -p(\Delta V) = 1.68 \times 10^5 (0.004 - 0.0135) \text{ J} = -1596 \text{ J}$$

PROBLEM 2

a)



b)

Determine T_B :

$$\frac{V_A}{V_B} = \left(\frac{T_B}{T_A}\right)^{\frac{5}{2}}$$

$$T_B = 10^{\frac{2}{5}} \times T_A = 754 \text{ K}$$

Determine T_C :

$$p_B V_B = nRT_B \text{ and } p_C V_C = nRT_C \text{ with } V_B = V_C$$

$$\frac{p_B}{p_C} = \frac{T_B}{T_C} = \frac{1}{5}$$

$$T_C = 5 \times T_B = 3770 \text{ K}$$

Determine T_D :

$$\frac{V_C}{V_D} = \frac{V_B}{V_A} = \frac{1}{10} = \left(\frac{T_D}{T_C}\right)^{\frac{5}{2}}$$

$$T_D = \left(\frac{1}{10}\right)^{\frac{2}{5}} T_C = 1500 \text{ K}$$

c)

No work done in steps 2 and 4 (no volume change).

Step 1:

$$w = C_V(T_A - T_B) = \frac{5}{2}R(300 - 754)\text{J} = -9.436 \text{ kJ}$$

Step 3:

$$w = C_V(T_C - T_D) = \frac{5}{2}R(3770 - 1500)\text{J} = 47.185 \text{ kJ}$$

d)

Efficiency can be increased by increasing the temperature range. Efficiency is lost, because in reality there is no such thing as a reversible process.

PROBLEM 3

a)

Thermal diffusion equation in steady state: $\frac{\partial T}{\partial t} = 0$,

i.e.
$$\frac{\kappa}{c} \nabla^2 T + \frac{H}{c} = 0 \text{ or } \frac{\kappa}{c} \nabla^2 T = -\frac{H}{c}$$

In 1D:
$$\frac{\partial^2 T}{\partial x^2} = -\frac{H}{\kappa}$$

Polynomial Ansatz:
$$T(x) = -\frac{1}{2} \frac{H}{\kappa} x^2 + \alpha x + \beta$$

b)

boundary condition 1:

$$T(0) = T_0$$

$$\beta = T_0$$

Boundary condition 2:

$$T(L) = T_0$$

$$-\frac{1}{2} \frac{H}{\kappa} L^2 + \alpha L + T_0 = T_0$$

$$\alpha = \frac{1}{2} \frac{H}{\kappa} L$$

Plug into the Ansatz:

$$T(x) = -\frac{1}{2} \frac{H}{\kappa} x^2 + \frac{1}{2} \frac{H}{\kappa} Lx + T_0$$

$$T(x) = \frac{H}{2\kappa} (Lx - x^2) + T_0$$

c)

The middle of the rod is at $x = \frac{L}{2}$.

Volume of the rod: $V = \pi r^2 L = \pi \times 10^{-6} \text{ m}^3$ i.e. $H = \frac{10}{\pi \times 10^{-6}} \text{ Wm}^{-3}$

$$T\left(\frac{L}{2}\right) = \frac{H}{2\kappa} \left(L \frac{L}{2} - \frac{L^2}{4} \right) + T_0 = \frac{H L^2}{2\kappa 4} + T_0$$

$$T\left(\frac{L}{2}\right) = \frac{10}{\pi \times 10^{-6}} \frac{1^2}{2 \times 50 \times 4} \text{ K} + 300 \text{ K} = 7958 + 300 \text{ K}$$

Way more than you would expect because losses (mostly EM radiation) are ignored.

PROBLEM 4

a)

$$Z_1 = \sum_{n=-1,0,1} e^{-\beta E_n} = e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}$$

b)

$$P_{-1} = \frac{e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}}$$

$$P_0 = \frac{1}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}}$$

$$P_1 = \frac{e^{-\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}}$$

Low temperature limit, $\beta\varepsilon \rightarrow \infty$ then

$$P_{-1} = \frac{e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} \rightarrow \frac{e^{\beta\varepsilon}}{e^{\beta\varepsilon}} \rightarrow 1$$

$$P_0 = \frac{1}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} \rightarrow \frac{1}{e^{\beta\varepsilon}} \rightarrow 0$$

$$P_1 = \frac{e^{-\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} \rightarrow \frac{0}{e^{\beta\varepsilon}} \rightarrow 0$$

High temperature limit, $\beta\varepsilon \rightarrow 0$ then

$$P_{-1} = \frac{e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} \rightarrow \frac{1}{1 + 1 + 1} = \frac{1}{3}$$

$$P_0 = \frac{1}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} \rightarrow \frac{1}{1 + 1 + 1} = \frac{1}{3}$$

$$P_1 = \frac{e^{-\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} \rightarrow \frac{1}{1 + 1 + 1} = \frac{1}{3}$$

c)

The mean energy of 1 particle is,

$$\langle \varepsilon \rangle = \sum_{n=-1,0,1} P_n E_n = \frac{-\varepsilon e^{\beta\varepsilon} + 0 + \varepsilon e^{-\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} = \varepsilon \frac{e^{-\beta\varepsilon} - e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}}$$

Because the particles are independent, we have,

$$U = N\langle \varepsilon \rangle = N\varepsilon \frac{e^{-\beta\varepsilon} - e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}}$$

Another method is through the N -particle partition function Z_N

$$Z_N = (Z_1)^N = (e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon})^N$$

And use

$$U = -\frac{\partial \ln Z_N}{\partial \beta} = -N \frac{\partial}{\partial \beta} (\ln(e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon})) \Rightarrow$$

$$U = -N \frac{\varepsilon e^{\beta\varepsilon} - \varepsilon e^{-\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} = N\varepsilon \frac{e^{-\beta\varepsilon} - e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}}$$

d)

Use $F = U - TS$ and $F = -kT \ln Z_N$ to find,

$$S = \frac{U - F}{T} = \frac{U + kT \ln Z_N}{T} \Rightarrow$$

$$S = N \frac{\varepsilon}{T} \frac{e^{-\beta\varepsilon} - e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} + k \ln(e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon})^N \Rightarrow$$

$$S = N \left\{ \frac{\varepsilon}{T} \frac{e^{-\beta\varepsilon} - e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} + k \ln(e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}) \right\}$$

At low temperatures $\beta \rightarrow \infty$ and we find:

$$\begin{aligned} S &\rightarrow N \left\{ \frac{\varepsilon}{T} \frac{0 - e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + 0} + k \ln(e^{\beta\varepsilon} + 1 + 0) \right\} \rightarrow N \left\{ \frac{\varepsilon}{T} \frac{-e^{\beta\varepsilon}}{e^{\beta\varepsilon}} + k \ln(e^{\beta\varepsilon}) \right\} \\ &= N \left\{ -\frac{\varepsilon}{T} + k\beta\varepsilon \right\} = N \left\{ -\frac{\varepsilon}{T} + \frac{k\varepsilon}{kT} \right\} = 0 \end{aligned}$$

According to the third law a system with a non-degenerate ground state has zero entropy. In this case all particles are in the $-\varepsilon$ state. There is only 1 such configuration, thus $S = k \ln 1 = 0$.

PROBLEM 5

a)

The partition function can be further evaluated as

$$Z_{ph} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots e^{-\beta(n_1\varepsilon_1+n_2\varepsilon_2+\dots)} = \prod_{i=1}^{\infty} \sum_{n_i=0}^{\infty} e^{-\beta(n_i\varepsilon_i)} = \prod_{i=1}^{\infty} \frac{1}{1 - e^{-\beta\varepsilon_i}}$$

where the geometrical series has been used.

Taking the logarithm gives:

$$\ln Z_{ph} = - \sum_{i=1}^{\infty} \ln(1 - e^{-\beta\varepsilon_i})$$

b)

For photons the momentum p is related to energy $\varepsilon = \hbar\omega = pc$. Using this and that the density of states for a *spinless* particle confined to an enclosure with volume V is (expressed as a function of the particle's momentum p):

$$g(p)dp = \frac{V}{h^3} 4\pi p^2 dp$$

in combination with the fact that the photon has two polarization states leads to,

$$g(\omega)d\omega = 2 \frac{V}{h^3} 4\pi \left(\frac{\hbar\omega}{c}\right)^2 d\left(\frac{\hbar\omega}{c}\right) = \frac{V}{\pi^2 \hbar^3} \left(\frac{\hbar}{c}\right)^3 \omega^2 d\omega = \frac{V\omega^2 d\omega}{\pi^2 c^3}$$

c)

Use the hint and make the energy levels continuous and replace the sum with an integral and use the density of states from a),

$$\ln Z_{ph} = - \int_0^{\infty} \ln(1 - e^{-\beta\hbar\omega}) g(\omega) d\omega = - \int_0^{\infty} \ln(1 - e^{-\beta\hbar\omega}) \frac{V\omega^2 d\omega}{\pi^2 c^3}$$

With the substitution $x = \beta\hbar\omega$ this leads to,

$$\ln Z_{ph} = - \frac{V}{\pi^2 c^3} \frac{1}{(\beta\hbar)^3} \int_0^{\infty} \ln(1 - e^{-x}) x^2 dx$$

Use the table with the integrals and constants,

$$\int_0^{\infty} \ln(1 - e^{-x}) x^2 dx = -\frac{\pi^4}{45}$$

To find:

$$\ln Z_{ph} = \frac{V}{\pi^2 c^3} \frac{1}{(\beta \hbar)^3} \frac{\pi^4}{45} = \frac{\pi^2 k^3}{45 \hbar^3 c^3} VT^3$$

d)

Helmholtz free energy F is given by: $F = -kT \ln Z_{ph}$

Thus,

$$F = -\frac{\pi^2 k^4}{45 \hbar^3 c^3} VT^4$$

From Helmholtz free energy $F = U - TS$ we find,

$$dF = dU - TdS - SdT = (TdS - PdV) - TdS - SdT = -PdV - SdT$$

Thus,

$$S = -\left(\frac{\partial F}{\partial T}\right)_V \text{ and } P = -\left(\frac{\partial F}{\partial V}\right)_T$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = \frac{4\pi^2 k^4}{45 \hbar^3 c^3} VT^3$$

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{\pi^2 k^4}{45 \hbar^3 c^3} T^4$$

$$U = -\frac{\pi^2 k^3}{45 \hbar^3 c^3} VT^4 + \frac{4\pi^2 k^3}{45 \hbar^3 c^3} VT^4 = \frac{\pi^2 k^4}{15 \hbar^3 c^3} VT^4$$

PROBLEM 6

a)

From the solution of the 2D-wave equation: $\varphi = A \sin k_x x \sin k_y y$ and taking this function to vanish at $x = y = 0$ and at $x = y = L$ results in,

$$k_x = \frac{n_x \pi}{L} \text{ and } k_y = \frac{n_y \pi}{L} \text{ with } n_x \text{ and } n_y \text{ non-zero positive integers.}$$

The total number of states with $|\vec{k}| < k$ is then given by, (the area of a quarter circle because we have only positive integers, with radius k divided by the area of the unit surface e.g. the surface of one state, in k -space).

$$\Gamma(k) = \frac{\frac{1}{4} \pi k^2}{\left(\frac{\pi}{L}\right)^2} = \frac{1}{4} \frac{L^2 k^2}{\pi}$$

The number of states between $k + dk$ and k is:

$$g(k)dk = \Gamma(k + dk) - \Gamma(k) = \frac{\partial \Gamma}{\partial k} dk = \frac{1}{2} \frac{L^2 k}{\pi} dk$$

Converting to energy $p = \sqrt{2mE} = \hbar k$ we find, $k = \frac{\sqrt{2mE}}{\hbar}$ and $dk = \frac{1}{2} \frac{2m}{\hbar \sqrt{2mE}} dE$

and accounting for the two spin states of the electron (extra factor of 2) we find

$$g(k)dk = 2 \times \frac{1}{2} \frac{L^2 k}{\pi} dk = \frac{mL^2}{\pi \hbar^2} dE = g(E)dE$$

b)

The Fermi energy is the value of the chemical potential μ at absolute zero temperature:

$$E_F = \mu(T = 0)$$

c)

Total number of fermions is given by,

$$N = \int_0^{\infty} n(E) g(E) dE$$

with,

$$n(E) = \frac{1}{e^{\beta(E-\mu)} + 1}$$

the mean number of fermions with energy E (Fermi-Dirac distribution)

Thus,

$$N = \frac{mL^2}{\pi\hbar^2} \int_0^{\infty} \frac{dE}{e^{\beta(E-\mu)} + 1}$$

And at $T = 0$, we have $E_F = \mu(T = 0)$ and thus, $n(E) = 1$ if $E < E_F$ and $n(E) = 0$ if $E > E_F$. Thus,

$$N = \frac{mL^2}{\pi\hbar^2} \int_0^{E_F} dE = \frac{mL^2}{\pi\hbar^2} E_F \Rightarrow E_F = N \frac{\pi\hbar^2}{mL^2}$$

d)

$$\begin{aligned} U &= \int_0^{\infty} E n(E) g(E) dE = \frac{mL^2}{\pi\hbar^2} \int_0^{\infty} \frac{E dE}{e^{\beta(E-\mu)} + 1} \xrightarrow{T=0} U = \frac{mL^2}{\pi\hbar^2} \int_0^{E_F} E dE = \frac{mL^2}{2\pi\hbar^2} E_F^2 \\ &= \frac{1}{2} N E_F \end{aligned}$$